ON THE PROBLEM OF THE ANALYTICAL DESIGN OF REGULATORS FOR DISTRIBUTED-PARAMETER SYSTEMS UNDER BOUNDARY-FUNCTION CONTROL

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The method of extension of a differential operator is carried over to systems of second-order parabolic linear differential equations. The problem of the analytical design of regulators with controls by boundary functions is reduced by means of the method of extension of a differential operator to a problem with distributed controls and is solved by the dynamic programming method.

In the theory of partial differential equations it is well known [1] that linear homogeneous equations with inhomogeneous boundary conditions are essentially equivalent to inhomogeneous equations with homogeneous boundary conditions. This can be shown by the method of extension of a differential operator [2-4].

Using delta-functions and their derivatives a linear homogeneous equation with inhomogeneous boundary conditions can be written as an inhomogeneous equation with homogeneous boundary conditions when certain continuity and differentiability conditions are fulfilled. In the case when the boundary conditions are the control functions, the inhomogeneous equation obtained can be treated as an optimal problem with distributed controls.

An analytical solution was obtained in [3, 4] by this method for the problem of bringing a rod's temperature upto a specified temperature distribution in a fixed interval of time with minimal energy, and an example with controls bounded in absolute value, analyzed earlier in [5], also was examined.

The analytical design problem for regulators for partial differential systems with distributed controls was considered in [6, 7]. Problems with boundary-function controls were studied in [7-9].

1. Let Ω be an open connected bounded subset of an m-dimensional Euclidean space, with the coordinate vector $\mathbf{s} = (s_1, \ldots, s_m)$. Let $\overline{\Omega}$ denote the closure of set Ω, and ω the boundary of Ω. The symbol $\mathbf{x} = \operatorname{col} \|x_1, \ldots, x_n\|$ will denote a column-matrix. We consider a controlled plant described by the following system of partial differential equations:

$$\partial \mathbf{u} / \partial t = L\mathbf{u}(\mathbf{s}, t) \qquad (\mathbf{s} \in \Omega, t > 0) \tag{1.1}$$

The initial conditions for system (1.1) have the form

$$\mathbf{u}(\mathbf{s}, 0) = \mathbf{u}_0(\mathbf{s}) \qquad (\mathbf{s} \in \overline{\Omega}, t = 0) \tag{1.2}$$

The boundary conditions may be of two kinds:

1° first boundary value problem

$$\mathbf{u}(\mathbf{s}, t) = \mathbf{f}_1(\mathbf{s}, t) \qquad (\mathbf{s} \in \omega, t > 0) \tag{1.3}$$

2° second boundary value problem

$$B\mathbf{u}(\mathbf{s}, t) = \mathbf{f}_2(\mathbf{s}, t) \qquad (\mathbf{s} \in \omega, t > 0) \tag{1.4}$$

Here $\mathbf{u}(\mathbf{s}, t)$ is the state vector of the system, $\mathbf{f}_\alpha(\mathbf{s}, t)$ $(\alpha = 1, 2)$ are boundary control functions, L and B are linear differential operators defined by the following relations:

$$\begin{gathered}
\mathbf{u} = \operatorname{col}\|u_1, \ldots, u_n\|, \qquad \mathbf{f}_\alpha = \operatorname{col}\|f_{\alpha 1}, \ldots, f_{\alpha n}\| \qquad (\alpha = 1, 2) \\
L\mathbf{u} = \operatorname{col}\|L_1\mathbf{u}, \ldots, L_n\mathbf{u}\|, \; B\mathbf{u} = \operatorname{col}\|B_1\mathbf{u}, \ldots, B_n\mathbf{u}\| \\
u_{it} = L_i\mathbf{u} = a_{ij}^{pq} u_{jpq} + a_{ij}{}^p u_{jp} + a_{ij} u_j \qquad (a_{ij}^{pq} = a_{ij}^{qp}) \\
B_i\mathbf{u} = \{a_{ij}^{pq} u_{jq} + \mu_{ij}(a_{ij}{}^p - (a_{ij}^{pq})_q u_j\} n_p \qquad (i = 1, \ldots, n) \\
u_{it} = \frac{\partial u_i}{\partial t}, \quad u_{jp} = \frac{\partial u_j}{\partial s_p}, \quad u_{jpq} = \frac{\partial^2 u_j}{\partial s_p \partial s_q}, \quad (a_{ij}^{pq})_q = \frac{\partial a_{ij}^{pq}}{\partial s_q}
\end{gathered} \tag{1.5}$$

In these relations n_p are the direction cosines of the outward normal to the boundary ω of region Ω; a_{ij}^{pq} are twice continuously differentiable functions, a_{ij}^p are continuously differentiable functions, a_{ij} and μ_{ij} are continuous functions of argument $\mathbf{s}$. In relations (1.5), as well as subsequently, pairs of like indices imply summation. Summation over the indices i, j, k, ν and μ is carried out from one to n, while over the indices p, q, l and ϑ from one to m. Summation is not carried out over indices which indicate the number of relations in the writing of the formulas (in the given case, i).

Those initial distributions $\mathbf{u}_0(\mathbf{s})$ for which system (1.1) with controls $\mathbf{f}_\alpha \equiv 0$ $(\alpha = 1, 2)$ has a unique solution twice-continuously differentiable in $\mathbf{s}$ are said to be admissible. The square matrix $Q(\mathbf{s}, \mathbf{s}') = \|Q_{ij}(\mathbf{s}, \mathbf{s}')\|_1^n$ symmetric with respect to $\mathbf{s}$ and $\mathbf{s}'$ is said to be positive definite if

$$\int_\Omega \int_\Omega v_i(\mathbf{s}) Q_{ij}(\mathbf{s}, \mathbf{s}') v_j(\mathbf{s}') \, d\Omega d\Omega' > 0 \qquad (\mathbf{v}(\mathbf{s}) \not\equiv 0)$$

for any continuous square-integrable vector-valued function $\mathbf{v}(\mathbf{s}) = \operatorname{col}\|v_1(\mathbf{s}), \ldots, v_n(\mathbf{s})\|$.

We assume that the boundary ω of region Ω can be split up into a finite number of $(m-1)$-dimensional hypersurfaces such that the tangent hyperplane to each of them varies continuously from point to point, and we suppose that we know the equation of boundary ω so that

$$\varphi(\mathbf{s}) = 0 \qquad (\mathbf{s} \in \omega) \tag{1.6}$$

In the case of a problem over an infinite time interval we are given the functional

$$J_\alpha = \int_0^\infty W_\alpha dt, \qquad W_\alpha = W^{(1)} + W_\alpha^{(2)} \qquad (\alpha = 1, 2) \tag{1.7}$$

In the case of a problem over a finite time interval we take the functional $J_\alpha{}^\tau$ in the following form:

$$J_\alpha{}^\tau = \int_0^\tau W_\alpha dt + W^{(3)} \qquad (\alpha = 1, 2) \tag{1.8}$$

Here

$$W^{(1)} = \int\limits_{\Omega}\int\limits_{\Omega} u_i(\mathbf{s}, t)\, Q_{ij}(\mathbf{s}, \mathbf{s}')\, u_j(\mathbf{s}', t)\, d\Omega\, d\Omega' \tag{1.9}$$

$$W_\alpha^{(2)} = \int\limits_{\omega} Q_j^{(\alpha)}(\mathbf{s})\, f_{\alpha j}^2(\mathbf{s}, t) d\omega \qquad (Q_j^{(\alpha)}(\mathbf{s}) > 0;\ \mathbf{s} \in \omega;\ j = 1, \ldots, n;\ \alpha = 1, 2)$$

$$W^{(3)} = \int\limits_{\Omega}\int\limits_{\Omega} u_i(\mathbf{s}, \tau)\, Q_{ij}^{\tau}(\mathbf{s}, \mathbf{s}')\, u_j(\mathbf{s}', \tau)\, d\Omega\, d\Omega'$$

The square matrices

$$Q(\mathbf{s}, \mathbf{s}') = \| Q_{ij}(\mathbf{s}, \mathbf{s}') \|_1^n, \qquad Q^{\tau}(\mathbf{s}, \mathbf{s}') = \| Q_{ij}^{\tau}(\mathbf{s}, \mathbf{s}') \|_1^n$$

are assumed to be positive definite, continuous and symmetric in $\mathbf{s}$ and $\mathbf{s}'$.

The functions $\mathbf{f}_\alpha(\mathbf{s}, t)$ $(\alpha = 1,2)$ continuous in $\mathbf{s}$ and t such that in the case of a problem over a finite time interval, $J_\alpha^{\tau} < \infty$, while in the case of a problem over an infinite time interval, $J_\alpha < \infty$ and the solution of system (1.1) is asymptotically stable in measure [6]

$$\rho = \left(\int\limits_{\Omega} \mathbf{u}'\mathbf{u} d\Omega \right)^{1/2}$$

are called admissible controls. Here $\mathbf{u}'$ is the vector transpose to the vector $\mathbf{u}$.

Let us assume that system (1.1) has a unique solution for any admissible initial distribution and any admissible control. The analytical design problem for regulators for system (1.1) with the functional J_α (J_α^{τ}) consists of seeking among the admissible controls the functions $\mathbf{f}_\alpha^{\circ} = \mathbf{f}_\alpha^{\circ}[\mathbf{u}]$ $(\alpha = 1,2)$, which yield the minimum of the functional J_α (J_α^{τ}) under any admissible initial distributions [6, 7]. The controls $\mathbf{f}_\alpha^{\circ}$ $(\alpha = 1,2)$, which solve the problem posed are said to be optimal.

2. The method of extension of a differential operator, as applied to the system (1.1)-(1.4) being considered, consists of the following [3]. We introduce an operator A and the operator A^* adjoint to it:

$$A(\cdot) = \left(\frac{\partial}{\partial t} - L \right)(\cdot), \qquad A^*(\cdot) = \left(-\frac{\partial}{\partial t} - L^* \right)(\cdot) \tag{2.1}$$

where L^* is the operator adjoint to operator L. The domain $D(A)$ of operator A is the set of all functions $\mathbf{u}(\mathbf{s}, t)$ satisfying the following conditions:

$$\begin{gathered} \mathbf{u} \in H, \quad A\mathbf{u} \in H, \quad \mathbf{u}(\mathbf{s}, 0) = 0, \quad \mathbf{s} \in \overline{\Omega} \\ \mathbf{u}(\mathbf{s}, t) = 0 \quad (B\mathbf{u}(\mathbf{s}, t) = 0), \quad \mathbf{s} \in \omega, \quad t > 0 \end{gathered} \tag{2.2}$$

where H is a Hilbert space. The operator A^* has the domain $D(A^*)$ which is the set of all functions $\mathbf{v}(\mathbf{s}, t)$, satisfying the following conditions:

$$\begin{gathered} \mathbf{v} \in H, \quad A^*\mathbf{v} \in H, \quad \mathbf{v}(\mathbf{s}, t) \to 0 \quad \text{for } t \to \infty,\ \mathbf{s} \in \overline{\Omega} \\ \mathbf{v}(\mathbf{s}, t) = 0 \quad (C\mathbf{v}(\mathbf{s}, t) = 0), \quad \mathbf{s} \in \omega, \quad t > 0 \end{gathered} \tag{2.3}$$

Here C is a boundary differential operator which for a given B can often be defined in such a way that the operators A and A^* are adjoint to each other. The explicit form of the operator C for B given by formula (1.5) will be determined below.

In the Hilbert space H we introduce the scalar product

$$(\mathbf{x}, \mathbf{y}) = \int_0^\infty \int_\Omega \bar{\mathbf{x}}' \mathbf{y} d\Omega dt \tag{2.4}$$

where $\bar{\mathbf{x}}$ denotes the complex conjugate of $\mathbf{x}$. For all $\mathbf{u} \in D\ (A)$ and all $\mathbf{v} \in D\ (A^*)$ the relation

$$(A^*\mathbf{v}, \mathbf{u}) = (\mathbf{v}, A\mathbf{u}) \tag{2.5}$$

is valid. The method of extension of a differential operator consists in the introduction of another operator A_e with a domain $D\ (A_e)$ broader than $D(A)$. Namely, as $D(A_e)$ we take the set of all functions $\mathbf{w}\ (\mathbf{s}, t)$ satisfying all the conditions imposed on $D\ (A)$, excepting that $\mathbf{w}\ (\mathbf{s},\ t)$ $(B\mathbf{w}\ (\mathbf{s},\ t))$ does not necessarily vanish when $\mathbf{s} \in \omega$. The scalar product $(A^*\ \mathbf{v},\ \mathbf{w})$ can be used to determine the operator A_e. We require that

$$(A^*\mathbf{v}, \mathbf{w}) = (\mathbf{v}, A_e\mathbf{w}) \tag{2.6}$$

By a Green's formula transformation and an integration by parts of the left-hand side of relation (2.6), we can obtain an expression for operator A_e, which contains delta functions and their derivatives.

Everything said above is vâlid also for the case of a finite time interval $[0,\ \tau]$, except for the obvious changes in the definition of scalar product (2.4) and in the third condition in (2.3).

The problem of the analytical design of regulators with boundary controls can be reduced with the aid of the method of extension of a differential operator to a problem with distributed controls. Here the homogeneous system (1.1) with nonzero (inhomogeneous) boundary conditions becomes an inhomogeneous system with zero (homogeneous) boundary conditions.

3. Using the realtion (2.5) for determining $A^*\mathbf{v}$, we have

$$A\mathbf{u} = \operatorname{col} \| A_1\mathbf{u}, \ldots, A_n\mathbf{u} \|$$

$$A_i\mathbf{u} = u_{it} - a_{ij}^{pq} u_{jpq} - a_{ij}{}^p u_{jp} - a_{ij} u_j \qquad (s \in \Omega,\ t > 0,\ i = 1, \ldots, n)$$

Let us assume for the moment that $\mathbf{u}_0\ (\mathbf{s}) \equiv 0$. By substituting $A\mathbf{u}$ into the right-hand side of relation (2.5) and by using Green's formula, an integration by parts, and the zero (homogeneous) boundary conditions, we obtain

$$(\mathbf{v}, A\mathbf{u}) = \int_0^\infty \int_\Omega v_i \cdot A_i \mathbf{u} d\Omega dt = \int_0^\infty \int_\Omega v_i (u_{it} - a_{ij}^{pq} u_{ipq} - a_{ij}^p u_{jp} - a_{ij} u_j) d\Omega dt =$$

$$= \int_0^\infty \int_\omega (u_j C_j \mathbf{v} + v_i B_i \mathbf{u}) d\omega dt + \int_0^\infty \int_\Omega A_j^* \mathbf{v} \cdot u_j d\Omega dt = \int_0^\infty \int_\Omega A_j^* \mathbf{v} \cdot u_j d\Omega dt = (A^*\mathbf{v}, \mathbf{u})$$

$$C\mathbf{v} = \operatorname{col} \| C_1\mathbf{v}, \ldots, C_n\mathbf{v} \|, \qquad A^*\mathbf{v} = \operatorname{col} \| A_1^*\mathbf{v}, \ldots, A_n^*\mathbf{v} \|$$

$$C_j\mathbf{v} = \{a_{ij}^{pq} v_{iq} + (\mu_{ij} - 1)(a_{ij}^p - (a_{ij}^{pq})_q) v_i\} n_p \qquad (j = 1, \ldots, n)$$

$$A_j^*\mathbf{v} = -v_{jt} - (a_{ij}^{pq} v_i)_{pq} + (a_{ij}^p v_i)_p - a_{ij} v_i \qquad (j = 1, \ldots, n) \tag{3.1}$$

Thus we have obtained explicit expressions for operators A^* and C.

Let $h\ (\mathbf{s})$ and $g\ (\mathbf{s})$ be continuously differentiable scalar functions.
The following formula is valid:

$$\int_\omega h_q(\mathbf{s})\, g(\mathbf{s})\, d\omega = -\int_\Omega h(\mathbf{s})\,(\delta_q(\varphi(\mathbf{s}))\, g(\mathbf{s}) + \delta(\varphi(\mathbf{s}))\, g_q(\mathbf{s}))\, d\Omega =$$

$$= -\int_\Omega h(\mathbf{s})\,(\delta(\varphi(\mathbf{s}))\, g(\mathbf{s}))_q\, d\Omega \tag{3.2}$$

Here, a subscript q denotes differentiation with respect to the argument s_q, summation over index q is understood, $\varphi(\mathbf{s})$ is the left-hand side of Eq. (1.6) defining boundary ω, $\delta(\varphi(\mathbf{s}))$ and $\delta_q(\varphi(\mathbf{s}))$ are delta functions defined by the relations

$$\int_\Omega \delta(\varphi(\mathbf{s}))\, g(\mathbf{s})\, d\Omega = \int_\omega g(\mathbf{s})\, d\omega, \quad \int_\Omega \delta_q(\varphi(\mathbf{s}))\, g(\mathbf{s})\, d\Omega = -\int_\omega g_q(\mathbf{s})\, d\omega \tag{3.3}$$

To determine an explicit expression for operator A_e we use relation (2.6). By substituting $A^* \mathbf{v}$ from (3.1) into the left-hand side of relation (2.6) and by using Green's formula and an integration by parts, after manipulations we obtain

$$(A^*\mathbf{v}, \mathbf{w}) = \int_0^\infty \int_\Omega (-v_{jt} - (a_{ij}^{pq} v_i)_{pq} + (a_{ij}^p v_i)_p - a_{ij} v_i)\, w_j\, d\Omega dt =$$

$$= \int_0^\infty \int_\omega (v_i B_i \mathbf{w} - w_j C_j \mathbf{v})\, d\omega dt + \int_0^\infty \int_\Omega v_i \cdot A_i \mathbf{w}\, d\Omega dt$$

Here we have assumed that $\mathbf{w}(\mathbf{s}, 0) \equiv 0$ $(\mathbf{s} \in \overline{\Omega})$. In the case of the first boundary value problem we use the boundary conditions

$$\mathbf{v}(\mathbf{s}, t) = 0, \quad \mathbf{w}(\mathbf{s}, t) = \mathbf{f}_1(\mathbf{s}, t) \quad (\mathbf{s} \in \omega,\ t > 0)$$

Formula (3.2) is obtained in the following form:

$$(A^*\mathbf{v}, \mathbf{w}) = -\int_0^\infty \int_\omega a_{ij}^{pq} f_{1j} v_{iq} n_p\, d\omega dt + \int_0^\infty \int_\Omega v_i \cdot A_i \mathbf{w}\, d\Omega dt =$$

$$= \int_0^\infty \int_\Omega v_i (A_i \mathbf{w} + h_i)\, d\Omega dt = (\mathbf{v}, A_e \mathbf{w})$$

$$\mathbf{h} = \operatorname{col} \| h_1, \ldots, h_n \|, \qquad h_i = (\delta(\varphi)\, a_{ij}^{pq} f_{1j} n_p)_q \qquad (i = 1, \ldots, n) \tag{3.4}$$

$$A_e \mathbf{w} = \operatorname{col} \| A_{e1} \mathbf{w}, \ldots, A_{en} \mathbf{w} \|, \qquad A_{ei} \mathbf{w} = A_i \mathbf{w} + h_i \qquad (i = 1, \ldots, n) \tag{3.5}$$

In the case of the second boundary value problem, by using the boundary condition

$$C\mathbf{v} = 0, \qquad B\mathbf{w} = \mathbf{f}_2 \qquad (\mathbf{s} \in \omega,\ t > 0)$$

we obtain

$$(A^*\mathbf{v}, \mathbf{w}) = \int_0^\infty \int_\omega v_i f_{2i}\, d\omega dt + \int_0^\infty \int_\Omega v_i A_i \mathbf{w}\, d\Omega dt = \int_0^\infty \int_\Omega v_i (A_i \mathbf{w} + \delta(\varphi) f_{2i})\, d\Omega dt =$$

$$= (\mathbf{v}, A_e \mathbf{w})$$

$$A_{ei} \mathbf{w} = A_i \mathbf{w} + \delta(\varphi) f_{2i} \qquad (i = 1, \ldots, n)$$

In the case of the first boundary value problem it is obvious that system (1.1)-(1.3) is equivalent to the system

$$\partial \mathbf{u}/\partial t = L\mathbf{u} - \mathbf{h} \qquad (\mathbf{s} \in \Omega,\ t > 0) \tag{3.6}$$

$$\mathbf{u}(\mathbf{s}, 0) = u_0(\mathbf{s}) \quad (\mathbf{s} \in \bar{\Omega},\ t = 0), \qquad \mathbf{u}(\mathbf{s}, t) = 0 \quad (\mathbf{s} \in \omega,\ t > 0)$$

In the case of the second boundary value problem the system (1.1),(1.2),(1.4) is equivalent to the system

$$\partial \mathbf{u}/\partial t = L\mathbf{u} - \delta(\varphi)\,\mathbf{f}_2 \qquad (\mathbf{s} \in \Omega,\ t > 0) \tag{3.7}$$

$$\mathbf{u}(\mathbf{s}, 0) = \mathbf{u}_0(\mathbf{s}) \quad (\mathbf{s} \in \bar{\Omega},\ t = 0), \qquad B\mathbf{u}(\mathbf{s}, t) = 0 \quad (\mathbf{s} \in \omega,\ t > 0)$$

We rewrite $W_\alpha^{(2)}$ $(\alpha = 1, 2)$ in the following form:

$$W_\alpha^{(2)} = \int_\Omega \delta(\varphi)\, Q_j^{(\alpha)} f_{\alpha j}^2 \, d\Omega \qquad (\alpha = 1, 2) \tag{3.8}$$

The modified form obtained for describing the original problem permits us to use the dynamic programming method for solving the analytical design problem for regulators with boundary controls in just the same way as in the case of a problem with distributed controls [6, 7].

Note 3.1. In the case when the boundary controls act only on a certain part of boundary ω, the function $\varphi(\mathbf{s})$ should be equal to zero on this part of the boundary and should not vanish outside it. In the case when different control functions are applied to different parts of the boundary, Eqs. (3.6) and (3.7) will contain several terms of form (3.4) in the right-hand sides.

4. We consider the analytical design problem for controllers over an infinite interval in the modified form obtained in the preceding section. Assuming that the principle of optimality [10] is valid, in accordance with the dynamic programming method we introduce the following functional:

$$\Pi_\alpha[\mathbf{u}(\mathbf{s}, t)] = \min_{f\alpha} \int_t^\infty W_\alpha d\eta \qquad (\alpha = 1, 2)$$

Having applied the formalism of the dynamic programming method, for the determination of $\Pi_\alpha[\mathbf{u}(\mathbf{s}, t)]$ we obtain the following functional equation:

$$\min_{f_\alpha} \left\{ W_\alpha + \delta_u \Pi_\alpha[\mathbf{u}(\mathbf{s}, t)] \frac{\partial \mathbf{u}}{\partial t} \right\} = 0 \qquad (\alpha = 1, 2) \tag{4.1}$$

Here the second term within the braces is the variational derivative of the functional Π_α with respect to $\mathbf{u}$ in the direction $\partial \mathbf{u} / \partial t$ (see [10] for example). Assume that the functional $\Pi_\alpha[\mathbf{u}(\mathbf{s}, t)]$ has the following form:

$$\Pi_\alpha[\mathbf{u}(\mathbf{s}, t)] = \int_\Omega \int_\Omega \mathbf{u}'(\mathbf{s}, t)^T P_\alpha(\mathbf{s}, \mathbf{s}')\, \mathbf{u}(\mathbf{s}', t)\, d\Omega\, d\Omega' \qquad (\alpha = 1, 2)$$

where $P_\alpha(\mathbf{s}, \mathbf{s}')$ $(\alpha = 1,2)$ is a positive-definite $(n \times n)$-matrix symmetric with respect to $\mathbf{s}$ and $\mathbf{s}'$ Since in what follows it will always be clear from the text which of the boundary value problems (first or second) we have in mind, for simplicity we drop the index α in the notations Π_α and P_α. Using Green's formula and the symmetry of the matrix $P(\mathbf{s}, \mathbf{s}')$, we obtain the following results:

1. First boundary value problem

$$\delta_u \Pi[\mathbf{u}(\mathbf{s},t)]\frac{\partial \mathbf{u}}{\partial t} = \int\limits_{\Omega}\int\limits_{\Omega}\left\{\left(\frac{\partial \mathbf{u}}{\partial t}\right)' P\mathbf{u} + \mathbf{u}'P\frac{\partial \mathbf{u}}{\partial t}\right\} d\Omega\, d\Omega' =$$

$$= \int\limits_{\Omega}\int\limits_{\Omega}\{(L\mathbf{u}-\mathbf{h})'P\mathbf{u} + \mathbf{u}'P(L\mathbf{u}-\mathbf{h})\}\, d\Omega\, d\Omega' = \tag{4.2}$$

$$= \int\limits_{\Omega}\int\limits_{\Omega}\{\mathbf{u}'L_{ss'}^{*}P\mathbf{u} - 2\mathbf{h}'P\mathbf{u}\}\, d\Omega\, d\Omega' + \int\limits_{\omega}\int\limits_{\Omega}\mathbf{g}_1'\mathbf{u}\, d\omega\, d\Omega' + \int\limits_{\Omega}\int\limits_{\omega}\mathbf{u}'\mathbf{g}_2\, d\Omega\, d\omega'$$

$$L_{ss'}^{*}P(\mathbf{s},\mathbf{s}') = L_s^{*}P(\mathbf{s},\mathbf{s}') + L_{s'}^{*}P(\mathbf{s},\mathbf{s}')$$

$$L_s^{*}P(\mathbf{s},\mathbf{s}') = \|L_{si}^{*}\mathbf{p}_j\|_1^n, \qquad L_{s'}^{*}P(\mathbf{s},\mathbf{s}') = \|L_{s'j}^{*}\mathbf{p}^i\|_1^n$$

$$\mathbf{p}_j = \|P_{1j},\ldots,P_{nj}\| \quad (j=1,\ldots,n), \qquad \mathbf{p}^i = \|P_{i1},\ldots,P_{in}\| \quad (i=1,\ldots,n)$$

$$L_{si}^{*}\mathbf{p}_j = (a_{ki}^{pq}P_{kj})_{pq} - (a_{ki}^{p}P_{kj})_p + a_{kj}P_{kj} \qquad (i,j=1,\ldots,n)$$

$$L_{s'j}^{*}\mathbf{p}^i = (a_{kj}^{pq}P_{ik})_{p'q'} - (a_{kj}^{p}P_{ik})_{p'} + a_{ki}P_{ik} \qquad (i,j=1,\ldots,n)$$

$$\mathbf{g}_1 = \mathrm{col}\,\|g_{11},\ldots,g_{1n}\|, \qquad \mathbf{g}_2 = \mathrm{col}\,\|g_{21},\ldots,g_{2n}\|$$

$$g_{1j} = \mathbf{p}_jB\mathbf{u} - \mathbf{u}'C\mathbf{p}_j \quad (j=1,\ldots,n) \qquad g_{2i} = \mathbf{p}^iB\mathbf{u} - \mathbf{u}'C\mathbf{p}^i \quad (i=1,\ldots,n)$$

$$C\mathbf{p}_j = \mathrm{col}\,\|C_1\mathbf{p}_j,\ldots,C_n\mathbf{p}_j\| \quad (j=1,\ldots,n), \quad C\mathbf{p}^i = \mathrm{col}\,\|C_1\mathbf{p}^i,\ldots,C_n\mathbf{p}^i\| \quad (i=1,\ldots,n)$$

$$C_i\mathbf{p}_j = \{a_{ki}^{pq}(P_{kj})_q + (\mu_{ki}-1)(a_{ki}^{p} - (a_{ki}^{pq})_q)P_{kj}\}\, n_p \qquad (i,j=1,\ldots,n)$$

$$C_j\mathbf{p}^i = \{a_{kj}^{pq}(P_{ik})_{q'} + (\mu_{ki}-1)(a_{kj}^{p} - (a_{kj}^{pq})_{q'}P_{ik}\}\, n_{p'} \qquad (i,j=1,\ldots,n)$$

$$(a_{ki}^{pq})_q = \partial a_{ki}^{pq}(\mathbf{s})/\partial s_q, \qquad (a_{kj}^{pq})_{q'} = \partial a_{kj}^{pq}(\mathbf{s}')/\partial s_a', \ldots$$

Using expressions (1.9) and (3.8) for $W_1^{(1)}$ and $W_1^{(2)}$, from (4.2) we get

$$\min_{f_1}\left\{\int\limits_{\Omega}\int\limits_{\Omega}(\mathbf{u}'(L_{ss'}^{*}P+Q)\mathbf{u} - 2\mathbf{h}'P\mathbf{u})\, d\Omega\, d\Omega' + \int\limits_{\Omega}\delta(\varphi)\, Q_j^{(1)}f_{1j}^2\, d\Omega + \right.$$

$$\left. + \int\limits_{\omega}\int\limits_{\Omega}\mathbf{g}_1'\mathbf{u}\, d\omega\, d\Omega' + \int\limits_{\Omega}\int\limits_{\omega}\mathbf{u}'\mathbf{g}_2\, d\Omega\, d\omega'\right\} = 0 \tag{4.3}$$

By substituting here $\mathbf{h}$ from (3.4) we find the function $\mathbf{f}_1^{\circ}$, which minimizes the left-hand side of equality (4.3). By omitting terms not containing function $\mathbf{f}_1$, in coordinate notation we have

$$\min_{f_{1j}}\left\{\int\limits_{\Omega}\delta(\varphi)\, Q_j^{(1)}f_{1j}^2\, d\Omega - 2\int\limits_{\Omega}\int\limits_{\Omega}(\delta(\varphi)\, a_{ij}^{pq}f_{1j}n_p)_q\, P_{ik}u_k\, d\Omega\, d\Omega'\right\} \qquad (j=1,\ldots,n)$$

By writing out in detail the derivative with respect to s_q and next using formula (3.2) and property (3.3) of delta function $\delta(\varphi)$, we obtain

$$\min_{f_{1j}}\left\{\int\limits_{\omega}Q_j^{(1)}f_{1j}^2\, d\omega + 2\int\limits_{\omega}a_{ij}^{pq}f_{1j}n_p\int\limits_{\Omega}(P_{ik})_q\, u_k\, d\Omega'\, d\omega\right\} \qquad (j=1,\ldots,n)$$

By equating the variational derivative with respect to f_1, to zero, we find the optimal control

$$f_{1j}^{\circ}(\mathbf{s},t) = -\frac{1}{Q_j^{(1)}(\mathbf{s})}\, a_{ij}^{pq}(\mathbf{s})\, n_p\int\limits_{\Omega}(P_{ik}(\mathbf{s},\mathbf{s}'))_q\, u_k(\mathbf{s}',t)\, d\Omega'$$

$$(\mathbf{s}\in\omega,\ t\geqslant 0,\ j=1,\ldots,n) \tag{4.4}$$

By substituting (4.4) into (4.3) and by prescribing the boundary conditions for the matrix $P(\mathbf{s},\mathbf{s}')$ in such a way that the two boundary integrals in (4.3) vanish, we obtain the following matrix equation for determining $P(\mathbf{s},\mathbf{s}')$:

$$L^*_{\mathbf{s}\mathbf{s}'}P + Q - P^{(1)} = 0 \quad (\mathbf{s}\in\Omega, \mathbf{s}'\in\Omega'), \qquad P^{(1)}(\mathbf{s},\mathbf{s}') = \| P_{ij}^{(1)}(\mathbf{s},\mathbf{s}') \|_1^n$$

$$P_{ij}^{(1)}(\mathbf{s},\mathbf{s}') = \int_\omega n_{p''} a_{k\mu}^{pq}(\mathbf{s}'')(P_{ik}(\mathbf{s},\mathbf{s}''))_{q''} \frac{1}{Q_\mu^{(1)}(\mathbf{s}'')} (P_{\nu j}(\mathbf{s}'',\mathbf{s}'))_{s''} a_{\nu\mu}^{ls}(\mathbf{s}'') n_{l''} d\omega''$$

$$P(\mathbf{s},\mathbf{s}') = 0 \quad (\mathbf{s}\in\omega, \mathbf{s}'\in\overline{\Omega}') \qquad P(\mathbf{s},\mathbf{s}') = 0 \quad (\mathbf{s}\in\overline{\Omega}, \mathbf{s}'\in\omega')$$

2. Second boundary value problem. Having carried out analogous calculations, we obtain the following equation for determining the matrix $P(\mathbf{s},\mathbf{s}')$:

$$L^*_{\mathbf{s}\mathbf{s}'}P + Q - P^{(2)} = 0 \quad (\mathbf{s}\in\Omega, \mathbf{s}'\in\Omega'), \qquad P^{(2)}(\mathbf{s},\mathbf{s}') = \| P_{ij}^{(2)}(\mathbf{s},\mathbf{s}') \|_1^n$$

$$P_{ij}^{(2)}(\mathbf{s},\mathbf{s}') = \int_\omega P_{i\mu}(\mathbf{s},\mathbf{s}'') \frac{1}{Q_\mu^{(2)}(\mathbf{s}'')} P_{\mu j}(\mathbf{s}'',\mathbf{s}') d\omega''$$

$$Cp_j = 0 \quad (\mathbf{s}\in\omega; \mathbf{s}'\in\overline{\Omega}'; j = 1,\ldots,n) \qquad Cp^i = 0 \quad (\mathbf{s}\in\overline{\Omega}; \mathbf{s}'\in\omega'; i = 1,\ldots,n)$$

The optimal control has the following form:

$$f_{2j}^\circ(\mathbf{s},t) = \frac{1}{Q_j^{(2)}(\mathbf{s})}\int_\Omega P_{jk}(\mathbf{s},\mathbf{s}') u_k(\mathbf{s}',t)\, d\Omega' \quad (\mathbf{s}\in\omega; t>0; j = 1,\ldots,n) \tag{4.5}$$

5. Let us consider the problem over a finite time interval. We introduce the following functional:

$$\Pi_\alpha[\mathbf{u}(\mathbf{s},t),t] = \min_{f_\alpha}\left\{\int_t^\tau W_\alpha d\eta + W^{(\alpha)}\right\} \quad (\alpha = 1,2) \tag{5.1}$$

For the determination of $\Pi_\alpha[\mathbf{u}(\mathbf{s},t),t]$ $(\alpha = 1,2)$ we obtain the equation

$$-\frac{\partial\Pi_\alpha}{\partial t} = \min_{f_\alpha}\left\{W_\alpha + \delta_u \Pi_\alpha[\mathbf{u}(\mathbf{s},t),t]\frac{\partial u}{\partial t}\right\} \quad (\alpha = 1,2)$$

Setting $t = \tau$ from (5.1) we get

$$\Pi_\alpha[\mathbf{u}(\mathbf{s},\tau)\tau] = W^{(\alpha)} \quad (\alpha = 1,2)$$

Assume that the solution of the functional equation has the form

$$\Pi_\alpha[\mathbf{u}(\mathbf{s},t),t] = \iint_{\Omega\Omega} \mathbf{u}'(\mathbf{s},t) P_\alpha(\mathbf{s},\mathbf{s}',t)\mathbf{u}(\mathbf{s}',t)\, d\Omega\, d\Omega' \quad (\alpha = 1,2)$$

where $P_\alpha(\mathbf{s},\mathbf{s}',t)$ $(\alpha = 1,2)$ is a square $(n\times n)$-matrix symmetric with respect to $\mathbf{s}$ and $\mathbf{s}'$. In just the same way as in the preceding section we find that the optimal controls $\mathbf{f}_1^\circ$ and $\mathbf{f}_2^\circ$ have the forms (4.4) and (4.5), respectively, on the interval $[0,\tau]$, while for the determination of the matrices $P_\alpha(\mathbf{s},\mathbf{s}',t)$ $(\alpha = 1,2)$ we obtain the following equations:

1. First boundary value problem.

$$-\partial P/\partial t = L^*_{ss'}P + Q - P^{(1)} \qquad (s\in\Omega,\ s'\in\Omega',\ t\in[0,\tau))$$
$$P(s,s',t)=0\ \ (s\in\omega,\ s'\in\bar\Omega',\ t\in[0,\tau)),\quad P(s,s',t)=0\ \ (s\in\bar\Omega,\ s'\in\omega',\ t\in[0,\tau))$$
$$P(s,s',\tau)=Q^\tau(s,s') \qquad (s\in\bar\Omega,\ s'\in\bar\Omega',\ t=\tau)$$

2. Second boundary value problem.

$$-\partial P/\partial t = L^*_{ss'}P + Q - P^{(2)} \qquad (s\in\Omega,\ s'\in\Omega',\ t\in[0,\tau))$$
$$Cp_j = 0 \quad (s\in\omega;\ s'\in\bar\Omega';\ t\in[0,\tau);\ j=1,\ldots,n)$$
$$Cp^i = 0 \quad (s\in\bar\Omega;\ s'\in\omega';\ t\in[0,\tau);\ i=1,\ldots,n)$$
$$P(s,s',\tau)=Q^\tau(s,s') \qquad (s\in\bar\Omega,\ s'\in\bar\Omega,\ t=\tau)$$

6. Example. Consider the problem of regulating a rod's temperature over a finite time interval:

$$\partial u/\partial t = a^2\partial^2u/\partial s^2 \qquad (0<s<1,\ 0<t\leqslant\tau) \tag{6.1}$$
$$u(s,0)=u_0(s)\quad (0\leqslant s\leqslant 1,\ t=0),\quad u(0,t)=0,\quad u(1,t)=f(t)\quad (0<t\leqslant\tau)$$

where $u(s, t)$ is the temperature mismatch, $f(t)$ is the control. We take the functional to be minimized in the form (1.8), where

$$W^{(1)}=\int_0^1\int_0^1 u(s,t)\,Q(s,s')\,u(s',t)\,ds\,ds', \qquad W^{(2)}=f^2(t)$$
$$W^{(3)}=\int_0^1\int_0^1 u(s,\tau)\,Q^\tau(s,s')\,u(s',\tau)\,ds\,ds'$$

Having determined the explicit form of operator A_s, we rewrite (6.1) in the following way:

$$\partial u/\partial t = a^2\frac{\partial^2u}{\partial s^2} - a^2\delta'(s-1)f(t)$$
$$u(s,0)=u_0(s),\ u(0,t)=u(1,t)=0$$

Here $\delta'(s-1)$ is the derivative with respect to s of the delta function $\delta(s-1)$. As before, the functional J^τ has the form (1.8), however, $W^{(2)}$ is rewritten as

$$W^{(2)}=f^2(t)\int_0^1\delta(s-1)\,ds$$

Assume that $\Pi[u(s,t),t]$ has the form

$$\Pi[u(s,t),t]=\int_0^1\int_0^1 u(s,t)\,P(s,s',t)\,u(s',t)\,ds\,ds'$$

With due regard to the zero boundary conditions for $P(s, s', t)$, the following equation:

$$\min_f\left\{\int_0^1\int_0^1 u\left(\frac{\partial P}{\partial t}+a^2\frac{\partial^2P}{\partial s^2}+a^2\frac{\partial^2P}{\partial s'^2}+Q\right)u\,ds\,ds' + f^2\int_0^1\delta(s-1)\,ds + 2a^2f\times\right.$$

$$\times \int_0^1 \int_0^1 \delta' (s - 1) \, Pudsds' \Big\} = 0 \tag{6.2}$$

corresponds to Eq. (4.3). The optimal control has the form

$$f^\circ (t) = a^2 \int_0^1 \frac{\partial P(s, s')}{\partial s} \bigg|_{s=1} u(s', t) \, ds' \tag{6.3}$$

Having substituted (6.3) into (6.2), we find the equation for $P(s, s', t)$:

$$- \frac{\partial P}{\partial t} = a^2 \left(\frac{\partial^2 P}{\partial s^2} + \frac{\partial^2 P}{\partial s'^2} \right) + Q - a^4 \frac{\partial P(s, s'', t)}{\partial s''} \frac{\partial P(s'', s', t)}{\partial s''} \bigg|_{s''=1}$$

$$(0 < s < 1, \ 0 < s' < 1, \ 0 \leqslant t < \tau)$$

$$P(s, s', t) = 0 \qquad (s = 0, 1; \ 0 \leqslant s' \leqslant 1; \quad 0 \leqslant t < \tau)$$

$$P(s, s', t) = 0 \qquad (0 \leqslant s \leqslant 1; \ s' = 0,1; \ 0 \leqslant t < \tau)$$

$$P(s, s', \tau) = Q^\tau (s, s') \qquad (0 \leqslant s \leqslant 1, \ 0 \leqslant s' \leqslant 1, \ t = \tau)$$

In the case being considered the operator $L_{ss'}^*$ has the form

$$L_{ss'}^* (\cdot) = a^2 \left(\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial s'^2} \right) (\cdot)$$

We seek $P(s, s', t)$ in the form of an expansion in the eigenfunctions of operator $L_{ss'}^*$:

$$P(s, s', t) = A_{\alpha\beta} (t) \sin \alpha\pi s \sin \beta\pi s' \tag{6.4}$$

Here and subsequently the summation with respect to the indices α, β, σ, and γ is carried out from one to ∞.

Let $Q(s, s')$ and $Q^\tau (s, s')$ have the following eigenfunction expansions:

$$Q(s, s') = q_{\alpha\beta} \sin \alpha\pi s \sin \beta\pi s', \qquad Q^\tau (s, s') = q_{\alpha\beta}^\tau \sin \alpha\pi s \sin \beta\pi s' \tag{6.5}$$

To determine the coefficients $A_{\alpha\beta} (t)$ we obtain the following system of ordinary Ricatti differential equations

$$- dA_{\alpha\beta}/dt = q_{\alpha\beta} - a^2\pi^2 (\alpha^2 + \beta^2) A_{\alpha\beta} - a^4 (-1)^{\sigma+\gamma} \sigma\gamma\pi^2 A_{\alpha\sigma} A_{\gamma\beta}$$

$$(0 \leqslant t < \tau; \ \alpha, \ \beta = 1,2,\ldots)$$

$$A_{\alpha\beta} (\tau) = q_{\alpha\beta}^\tau \qquad (t = \tau; \ \alpha, \beta = 1, 2, \ldots)$$

For practical purposes we restrict ourselves to a finite number of terms in expansions (6.4), (6.5). In the case when we restrict ourselves to just one term in expansions (6.4), (6.5), we obtain the following equation for the determination of $A_{11}(t)$:

$$-dA_{11}/dt = q_{11} - 2a^2\pi^2 A_{11} - a^4\pi^2 A_{11}^2, \qquad A_{11}(\tau) = q_{11}^\tau$$

Its solution has the form

$$A_{11} (t) = b + \frac{1}{b_1 + b_2 e^{\psi(t)}}, \qquad b = - \frac{1 - \sqrt{1 + q_{11}\pi^{-2}}}{a^2}$$

$$b_1 = - \frac{a^2}{2(1 + a^2 b)}, \qquad b_2 = \frac{1}{q_{11}^\tau - b} - b^1, \qquad \psi (t) = 2\pi^2 a^2 (1 + a^2 b)(\tau - t)$$

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